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Tachyon condensation on torus and T-duality

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Abstract: We find an exact solution, with a nonzero net D-brane charge, in the boundary string field theory of brane-anti-brane pairs on a torus. We explicitly take the T-dual of this configuration. The Nahm-transformation of the instantons is derived from the tachyon condensation.

Keywords: Tachyon Condensation, D-branes, Solitons Monopoles and Instantons, String Field Theory.

Contents

1. Introduction

The open sting tachyon condensation on unstable D-branes have been intensively investigated in the past decade [1, 2]. Among them, the exact solutions of the tachyon condensation, were found in the boundary string field theory[1] [3 − 7] or in the boundary state [8] on $\mathbf{R}^r$.[2] Those solutions include the topologically non-trivial solutions, for example, the kink or the vortex, which represent lower dimensional D-branes, with codimension one and two, respectively. This constructions of lower dimensional D-branes from unstable D-branes by the tachyon condensation were known as the decent relations [1]. On the other hand, we can construct higher dimensional D-branes from lower dimensional unstable D-branes, like the matrix models, by the tachyon condensation on $\mathbf{R}^r$ [12 − 14], which were known as the ascent relations.

Since $\mathbf{R}^r$ is topologically trivial and non-compact, there is no winding modes and the solutions in the boundary string field theory can have a trivial bundle. Since the torus is simplest non trivial compact manifold, the study of the tachyon condensation on torus will be interesting.[3] However, it will be very difficult to find an exact solution of a soliton on a D$-\overline{\text{D}}$-brane pair on a torus. To be explicit, let us consider a vortex soliton of the tachyon of a D2$-\overline{\text{D2}}$-brane pair on T^2. This soliton will represent a D0-brane. Since the torus is an orbifold of $\mathbf{R}^2$, it seems easy to construct such soliton, however, the orbifolding of the

[1] Recently, the exact solution in the Witten's cubic string field theory for the bosonic string was found in [9]. it may represent a closed string vacuum.

[2] Recently, the boundary string field theory was reconstructed via the boundary state [10, 11].

[3] On a torus with the self dual radius, an exact solution of the tachyon condensation of a D$-\overline{\text{D}}$-brane pairs was given in [1, 15] by the marginal deformations (the "tachyon" is massless). This solution corresponds to the lower dimensional D$-\overline{\text{D}}$-brane systems. By the marginal deformation, we always has a D$-\overline{\text{D}}$-brane or non BPS D-branes which do not have net D-brane charges because of the charge conservation. In this paper we will study the soliton with a net D-brane charge.

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solution is not straightforward. Actually, on $\mathbf{R}^2$ the D0-brane solution is represented by a following non-periodic configuration:

$$\begin{aligned} T &= u(\xi^1 + i\xi^2), \quad (u \to \infty), \\ A_\mu^{(1)} &= A_\mu^{(2)} = 0, \qquad (\mu = 1, 2), \end{aligned} \tag{1.1}$$

in the boundary string field theory. Here T is the tachyon and $A_\mu^{(1)}$ and $A_\mu^{(2)}$ are the gauge fields on the D2-brane and anti-D2-brane, respectively. This is the exact solution of the equations of motion in the $u \to \infty$ limit. Obviously, it is difficult to extend this solution (1.1) on $\mathbf{R}^2$ to a solution on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ because of the non periodicity of (1.1).[4] Moreover, the gauge fields can not be trivial on T^2 because it is a compact space. It will be interesting, but, highly non-trivial to construct a vortex solution ($\mathcal{D}_\mu T \sim 0$) with non-zero $A_\mu^{(i)}$.

In this paper, we consider the tachyon condensation of D$-\overline{\text{D}}$-brane pairs on a torus and find exact solutions, which have a net D-brane charge, in the boundary string field theory or the boundary state formalism. Our construction uses infinitely many D$-\overline{\text{D}}$-brane pairs instead of a pair. By the T-dual transformation, we can change the dimensions of the D$-\overline{\text{D}}$-brane pairs and we will see that the soliton on the D0$-\overline{\text{D}}$-brane pairs which represents a D($2p$)-brane is the simplest one.

As an application of the soliton solution, we can consider the Nahm-transformation [16, 17] which maps an anti-self dual gauge field (instanton) of U(N) with the instanton number k on four dimensional torus T^4 to an anti-self dual configuration of U(k) with the instanton number N on on a dual torus $\tilde{T}^4$. In string theory, the bound state of N D($p+4$)-branes and k Dp-branes on T^4 is given by the U(N) gauge instanton. If we introduce a probe D($p-4$)-brane and consider the low energy limit on it, the Nahm transformation was interpreted as the T-dual transformation [18], generalizing the ADHM(N) cases [19, 20]. Recently, the ADHM transformation was given in a D-brane setup without a probe D-brane and nor a low energy limit [21, 22] by the method using the tachyon condensation [23, 24]. For a bound state of two different D-branes, say the D($p+q$)-branes and the Dp-branes, this method gives the equivalence between the descriptions using the D($p+q$)-branes and using the Dp-branes. This method can be applied to T^4 case and we will see that the Nahm transformation is naturally interpreted as this equivalence (plus the T-dual transformation). It is worth noticing that this equivalence is exact in α', therefore, the N D4-brane with the k instanton on T^4 is equivalent to k D4-branes with the N instanton on the T-dual torus $\tilde{T}^4$, which has a sub-stringy size if the size of T^4 is much bigger than string scale.[5]

[4] If we take the bundle on the D2-brane as (2.7) and the trivial bundle on the anti-D2-brane, a general tachyon field would be written as

$$T(\xi^1, \xi^2) = u\left(\sum_{n\in\mathbf{Z}} H\left(n + \frac{\xi^1}{2\pi L_1}\right) e^{i\frac{\xi^2}{L_2}n}\right) G(x^2, x^2), \tag{1.2}$$

where $G(\xi^1, \xi^2)$ is a periodic function of ξ^μ.

[5] We assume we can employ off-shell boundary states, which are naive extensions of the boundary state, as in [21, 22]. They have possibility of suffering from divergences when away from on-shell background fields.

This paper is organized as follows. In section 2, we review how to obtain the T-dual picture of the D0-branes on torus, according to [25]. In section 3, we give an exact solution in the boundary string field theory of brane-anti-brane pairs on torus. We take the T-dual of this configuration. The Nahm-transformation of the instantons is derived from the tachyon condensation. We conclude with some discussions in section 4.

2. D0-branes on torus and T-dual

In this section, we will review how to describe D0-branes in type II superstring theory on a (rectangular) torus as an orbifold $T^r = \mathbf{R}^r/\mathbf{Z}^r$, whose periodic coordinates $0 \leq x^\mu < 2\pi R_\mu$, according to [25, 26] and how to take the T-dual of the D0-branes [25].

We consider only the scalars corresponding to the locations of the D0-branes in the torus, $X^\mu(t)$, where $\mu = 1, \ldots, r$ although there are many fields on the D0-branes. In this paper, the time t is always fixed, and thus abbreviated below. If we consider the time-independent X^μ, it can be considered as the static configuration in the $A_0 = 0$ gauge.

Since $T^r = \mathbf{R}^r/\mathbf{Z}^r$, the N D-branes on tours will be equivalent to the $N \times \infty$ D-branes on $\mathbf{R}^r$ whose coordinates X^μ will be operator valued Hermite $N \times N$ matrices. Here we regard an $\infty \times \infty$ matrix as an operator.

By the orbifolding, we need the following identification with translation operators U'_ν along x^ν which should be operator valued $N \times N$ unitary matrices:

$$U'_\nu X^\mu {U'_\nu}^{-1} = \Omega'_\nu \left(X^\mu + \delta_{\mu\nu} 2\pi R_\nu 1_N\right) {\Omega'_\nu}^{-1}, \tag{2.1}$$

where Ω'_ν is an $N \times N$ unitary matrix, i.e. a gauge transformation of the N D-branes. Throughout this paper, we take a convention that an index ν is not summed over except explicitly indicated by $\sum_\nu$. We will define $U_\nu = {\Omega'_\nu}^{-1} U'_\nu$, then

$$U_\nu X^\mu {U_\nu}^{-1} = X^\mu + \delta_{\mu\nu} 2\pi R_\nu 1_N. \tag{2.2}$$

A representation of (2.2) is

$$\begin{aligned} X^\mu &= 2\pi\alpha' \left(i\frac{\partial}{\partial \xi^\mu} + \tilde{A}_\mu(\xi) \right) \\ U_\nu &= e^{i\frac{\xi^\nu}{L_\nu}}, \end{aligned} \tag{2.3}$$

where $\tilde{A}_\mu(\xi)$ is an $N \times N$ matrix and

$$L_\nu = \frac{\alpha'}{R_\nu}. \tag{2.4}$$

Here ξ_ν is the periodic coordinate of the T-dual torus $\tilde{T}^r$ and $0 \leq \xi_\nu < 2\pi L_\nu$ and $\tilde{A}_\mu$ is the gauge field of the N Dr-branes on $\tilde{T}^r$.[6] Note that this implies that the gauge field $\tilde{A}_\mu(\xi)$

However, the off-shell boundary states have a natural interpretation in consistency with the boundary string field theories. Furthermore, our main concern is the on-shell configurations although finding those are not discussed in this paper. Actually, the instanton configurations on torus is expected to be on-shell for all order in α' as discussed in [21, 22].

[6] The gauge transformation of the N D0-branes should not change the (2.2). Thus the transformation is generated by a $N \times N$ unitary matrix $U_{N\times N}(\xi, \frac{\partial}{\partial \xi})$ which commutes with U_ν. This is actually a gauge transformation of the N Dr-branes, i.e. a unitary matrix $U_{N\times N}(\xi)$.

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is a connection on the $\tilde{T}^r$, whose component is not necessary a periodic function of ξ. If the bundle of the N Dr-branes on $\tilde{T}^r$ with the gauge field $\tilde{A}_\mu(\xi)$ is trivial, i.e. $\tilde{A}_\mu(\xi)$ is periodic, the base of the Hilbert space is spanned by

$$e^{-i\sum_{\nu=1}^{r}\left(\frac{\xi^\nu n_\nu}{L_\nu}\right)}v_N, \tag{2.5}$$

where $n_\nu \in \mathbf{Z}$ and v_N is a base of a N-vector. If the bundle of Dr-branes is non-trivial, the base of the Hilbert space will be the sections of the bundle on the dual torus $\tilde{T}^r$.

Finally, let us consider a bound state of a D0-brane and m Dr-branes on the torus T^r. First, we sketch how to construct a Dr-brane within m D0-branes on the T-dual torus $\tilde{T}^r$. We will consider $r = 2$ case as an example. The bound state of the D2-brane and the D0-branes on $\tilde{T}^2$ will be given by

$$\begin{aligned} \tilde{A}_1 &= 0,\ \tilde{A}_2 = \tilde{F}\xi^1, \\ \tilde{F} &= \frac{m}{2\pi L_1 L_2}, \end{aligned} \tag{2.6}$$

where m is an integer which is the D0-brane charge. The transition function (or the gauge transformation) between the different patches is given by $\tilde{A}_\mu(\xi^1 + 2\pi L_1, \xi^2) = \tilde{\Omega}_1 \tilde{A}_\mu \tilde{\Omega}_1^{-1} - i(\partial_\mu \tilde{\Omega}_1)\tilde{\Omega}_1^{-1}$ and $\tilde{A}_\mu(\xi^1, \xi^2 + 2\pi L_2) = \tilde{\Omega}_2 \tilde{A}_\mu \tilde{\Omega}_2^{-1} - i(\partial_\mu \tilde{\Omega}_2)\tilde{\Omega}_2^{-1}$ where

$$\tilde{\Omega}_1 = e^{i\frac{\xi^2}{L_2}}, \quad \tilde{\Omega}_2 = 1. \tag{2.7}$$

Note that this is the exact solution. Here an exact solution means that a solution of the equations of motions of the D2-brane (string field theory) action including all order in the α' expansions, but leading order in the string coupling g_s. Then, from the T-dual map (2.3), we can read the D0-brane configuration X^μ of the bound state of the D0-brane and the m Dr-branes on T^r.

3. D0$-\overline{\text{D0}}$ pairs on torus

In this section, we will construct a solution which is equivalent to M D($2p$)-branes in the boundary string field theory of infinitely many D0$-\overline{\text{D0}}$-brane pairs on a torus T^{2p}.

First, we consider the infinitely many D0$-\overline{\text{D0}}$-brane pairs on $\mathbf{R}^{2p}$. The solution which is equivalent to the M D($2p$)-branes $\mathbf{R}^{2p}$ with gauge field $A_\mu(x)$ is

$$\begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix} = \lim_{u\to\infty} u\, \Gamma^\mu \otimes (1_{M\times M} \otimes \hat{p}_\mu - A_\mu(\hat{x})), \qquad X^\mu = 1_{2^p\times 2^p} \otimes 1_{M\times M} \otimes \hat{x}^\mu, \tag{3.1}$$

where X^μ is the transverse scalars of D0-branes and the T is the tachyon which acts on the D0-branes and $T^\dagger$ acts on the anti-D0-branes, which correspond to the anti-chiral spinors. Here we have set that the anti-D0-branes has the transverse scalars with the same v.e.v as the D0-branes. The operators $\hat{x}^\mu, \hat{p}_\mu$ satisfy $[\hat{x}^\mu, \hat{p}_\nu] = i\delta_{\mu,\nu}$ and Γ^μ is the Dirac gamma matrix of SO($2p$) which satisfies $\Gamma \equiv i^{-p}\Gamma^1\Gamma^2\cdots\Gamma^{2p} = \begin{pmatrix} 1_{2^{p-1}\times 2^{p-1}} & 0 \\ 0 & -1_{2^{p-1}\times 2^{p-1}} \end{pmatrix}$. Note that T and X^μ act on the Dirac spinors which transformed as a fundamental representation of the U(M) gauge symmetry on the manifold spanned by M D($2p$)-branes. Thus the (3.1) can be written as

$$\begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix} = \lim_{u\to\infty} u\, \not{D}, \qquad X^\mu = x^\mu. \tag{3.2}$$

Using this configuration we can construct the M D($2p$)-branes with gauge field $A_\mu(x)$ on the torus T^{2p}, which is spanned by $0 \le x^\mu \le 2\pi R_\mu$, by the orbifolding of $\mathbf{R}^{2p}$. Here we assume that $A_\mu(x)$ satisfies

$$A_\rho(x_\mu + \delta^{\mu\nu} 2\pi R_\nu) = \Omega_\nu A_\rho(x)\Omega_\nu^{-1} - i\Omega_\nu \partial_\rho \Omega_\nu^{-1}, \tag{3.3}$$

where Ω_ν is a transition function (or gauge transformation) on the torus. Thus the A_μ is a gauge field on $\mathbf{R}^{2p}$ which is extended from the gauge field on the torus, i.e. a pull back connection of the map from $\mathbf{R}^{2p}$ to $T^{2p} = \mathbf{R}^{2p}/\mathbf{Z}_{2p}$. The constraint for the tachyon T by the orbifolding may be same as the constraint for transverse coordinates. Thus we require that

$$\begin{aligned} U_\nu X^\mu U_\nu^{-1} &= X^\mu + \delta_{\mu\nu} 2\pi R_\nu 1_N, \\ U_\nu T U_\nu^{-1} &= T, \end{aligned} \tag{3.4}$$

where we take same Ω_ν in (2.1) for the D0-branes and the anti-D0-branes. Then the configuration (3.1) is consistent with the constraint (3.4) of the orbifolding if we take

$$U_\nu = \Omega_\nu\, e^{2\pi i \hat{p}_\nu R_\nu}. \tag{3.5}$$

This is obvious if we notice that the D0$-\overline{\text{D0}}$-branes given by the configuration (3.1) uniformly distributed in $\mathbf{R}^{2p}$ and the unit shift (3.5) is a symmetry. Note that

$$U_\rho U_\mu = U_\mu U_\rho, \tag{3.6}$$

which is from the fundamental property of the transition functions.

Therefore, the configuration (3.1) with the orbifolding operator (3.5) is a consistent configuration of the infinitely many D0$-\overline{\text{D0}}$-brane pairs on a torus T^{2p} which is equivalent to M D($2p$)-branes with the gauge field $A_\mu(x)$. Since the orbifolding will consistently truncate the equations of motion or the (on-shell) boundary state, (3.1) with the orbifolding by the generator (3.5) will be an exact solution on the torus if we set $A_\mu = 0$ or, for example, an anti self-dual configuration for $p = 2$.[7]

3.1 Nahm transformation and tachyon condensation on D0$-\overline{\text{D0}}$ pairs

If we consider M D(2p)-branes with a nontrivial gauge bundle on the torus, it is the bound state of M D(2p)-branes and the lower dimensional D-branes, for example D0-branes. In this case, following [23] (see also [27]) we can find a configuration of D0-branes which is equivalent to the bound state. We will apply this to the solution on the torus and see that the Nahm transformation naturally appears.

[7]It is desirable and interesting to study the solution on the torus in the boundary state formalism.

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In [23], we first take a configuration of D0$-\overline{\text{D0}}$ pairs which represents the M D(2p)-branes by the tachyon condensation. Then the tachyon is diagonalized by the gauge transformation and then only D0-branes which corresponds to zero modes remain after the tachyon condensation, namely the $u \to \infty$ limit. Then we see that the remaining D0-branes have the transverse scalars or the matrix coordinate $\bar{X}^\mu$ is given by just a truncation of the Chan-Patton-Hilbert space to those composed by the the zero modes only:

$$\left(\bar{X}^\mu\right)^i{}_j = \langle\, i \,|\, X^\mu \,|\, j \,\rangle\,, \tag{3.7}$$

where $|\, i \,\rangle$ is a zero mode of the tachyon. This gives the D0-brane picture of the boundary state.

For the M D(2p)-branes with a nontrivial gauge bundle on the torus, the tachyon,

$$\begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix} = \lim_{u\to\infty} u\, \not{D}, \tag{3.8}$$

acts on $\Psi(x)$ which is a spinor on $\mathbf{R}^{2p}$. Now we decompose a spinor on $\mathbf{R}^{2p}$ into a spinor on torus and a plain wave like the Bloch wave function:

$$\Psi_\xi(x) = e^{i\frac{1}{2\pi\alpha'}\xi_\mu x^\mu}\psi_\xi(x) \tag{3.9}$$

where $\psi_\xi(x)$ is

$$U_\nu \psi_\xi(x) = \psi_\xi(x), \tag{3.10}$$

which means that $\psi_\xi(x)$ is a section of the spinor bundle on T^{2p} and $0 \leq \xi_\mu < 2\pi L_\mu$. Indeed, Ψ_ξ is the eigen state of the unitary operator U_ν:

$$U_\nu \Psi_\xi(x) = e^{i\frac{\xi_\nu R_\nu}{\alpha'}}\Psi_\xi(x). \tag{3.11}$$

Thus any spinor $\Psi(x)$ on $\mathbf{R}^{2p}$ can be written as

$$\Psi(x) = \int_0^{\frac{2\pi\alpha'}{R_1}} d\xi_1 \int_0^{\frac{2\pi\alpha'}{R_2}} d\xi_2 \cdots \int_0^{\frac{2\pi\alpha'}{R_{2p}}} d\xi_{2p} e^{i\frac{1}{2\pi\alpha'}\xi_\mu x^\mu}\; \psi_\xi(x), \tag{3.12}$$

because any eigen state of U_ν can be written as (3.9). Using this, we have

$$\not{D}\Psi(x) = \int_0^{\frac{2\pi\alpha'}{R_1}} d\xi_1 \int_0^{\frac{2\pi\alpha'}{R_2}} d\xi_2 \cdots \int_0^{\frac{2\pi\alpha'}{R_{2p}}} d\xi_{2p} e^{i\frac{1}{2\pi\alpha'}\xi_\mu x^\mu}\; \not{D}_\xi \psi_\xi(x), \tag{3.13}$$

where

$$\not{D}_\xi = \Gamma^\mu\left(\hat{p}_\mu - A_\mu(\hat{x}) + \frac{\xi_\mu}{2\pi\alpha'}\right). \tag{3.14}$$

Then, the zero modes of the tachyon, $\not{D}\Psi(x) = 0$, is given by

$$\Psi^i_\xi(x) = e^{i\frac{1}{2\pi\alpha'}\xi_\mu x^\mu}\psi^i_\xi(x) \tag{3.15}$$

where $\psi^i_\xi(x)$ is a zero mode of $\not{D}_\xi$, i.e. it satisfies

$$\not{D}_\xi \psi^i_\xi(x) = 0, \tag{3.16}$$

$(i = 1, \cdots, m)$ and m is the number of the zero modes of $\not{D}_\xi$.[8] From the index theorem [28], we know that m does not depend on ξ. For $p = 2$, m is the instanton number. The zero modes are labeled by ξ and i. We will see that the discrete eigen values of $\frac{\partial}{\partial \xi^\mu}$ parameterize mirror images of D0-branes by the $\mathbf{Z}^{2p}$ orbifolding.

We normalize the zero modes as

$$\frac{1}{2\pi} \int_{\mathbf{R}^{2p}} d^{2p}x \, \Psi_\xi^i(x)^\dagger \, \Psi_{\xi'}^j(x) = \delta(\xi - \xi')\delta_{ij}, \tag{3.17}$$

which is equivalent to

$$\int_{T^{2p}} d^{2p}x' \, \psi_\xi^i(x')^\dagger \, \psi_\xi^j(x') = \delta_{ij}, \tag{3.18}$$

where $0 \leq x'^\nu \leq 2\pi R_\nu$. (Because of the Euclidean nature, $\Psi^\dagger \Psi$ is the SO$(2p)$ invariant.) This can been seen from

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\mathbf{R}^{2p}} d^{2p}x \, \Psi_\xi^i(x)^\dagger \, \Psi_{\xi'}^j(x) \\
&\quad = \frac{1}{2\pi} \int_{T^{2p}} d^{2p}x' \sum_{l_1, \cdots, l_{2p} \in \mathbf{Z}} \exp\left(i \frac{(x'^\mu + 2\pi R_\mu l_\mu)(\xi^\mu - \xi'^\mu)}{2\pi\alpha'} \right) \psi_\xi^i(x)^\dagger \, \psi_{\xi'}^j(x) \\
&\quad = \int_{T^{2p}} d^{2p}x' \, \psi_\xi^i(x')^\dagger \, \psi_{\xi'}^j(x') 2\pi\delta(\xi - \xi'),
\end{aligned} \tag{3.19}$$

where $x^\nu = x'^\nu + 2\pi R_\nu l_\nu$ and we have used $\psi_\xi^i(x) = \Omega_\nu^{-1} \psi_\xi^i(x')$ which implies $\psi_\xi^i(x)^\dagger \, \psi_{\xi'}^j(x) = \psi_\xi^i(x')^\dagger \, \psi_{\xi'}^j(x')$.

Now we can evaluate the coordinate $\bar{X}^\mu$ of m D0-branes corresponding to the remaining m zero modes:

$$\begin{aligned}
\left(\bar{X}^\mu\right)^{i,\xi}{}_{j,\xi'} &= \frac{1}{2\pi} \int_{\mathbf{R}^{2p}} d^{2p}x \, \Psi_\xi^i(x)^\dagger \, x^\mu \, \Psi_{\xi'}^j(x) \\
&= \alpha' \int_{\mathbf{R}^{2p}} d^{2p}x \, \Psi_\xi^i(x)^\dagger \left(-i \frac{\partial}{\partial \xi'^\mu} \Psi_{\xi'}^j(x) + i e^{i \frac{1}{2\pi\alpha'} \xi'_\mu x^\mu} \frac{\partial}{\partial \xi'^\mu} \psi_{\xi'}^j(x) \right) \\
&= 2\pi\alpha' \left(i\delta_{ij} \frac{\partial}{\partial \xi^\mu} + \left(\tilde{A}_\mu(\xi)\right)^i{}_j \right) \delta(\xi - \xi'),
\end{aligned} \tag{3.20}$$

where

$$\left(\tilde{A}_\mu(\xi)\right)^i{}_j = i \int_{T^{2p}} d^{2p}x' \, \psi_\xi^i(x')^\dagger \, \frac{\partial}{\partial \xi^\mu} \psi_\xi^j(x'). \tag{3.21}$$

This means that

$$\langle \xi, i \,|\, \bar{X}^\mu = 2\pi\alpha' \left(i\delta_{ij} \frac{\partial}{\partial \xi^\mu} + \left(\tilde{A}_\mu(\xi)\right)^i{}_j \right) \langle \xi, j \,|, \tag{3.22}$$

thus $\bar{X}^\mu = 2\pi\alpha' \left(i\delta_{ij} \frac{\partial}{\partial \xi^\mu} + \left(\tilde{A}_\mu(\xi)\right)^i{}_j \right)$ in this basis. Moreover, from

$$U_\nu \Psi_\xi^i(x) = e^{i \frac{\xi^\nu}{L_\nu}} \Psi_\xi^i(x), \tag{3.23}$$

[8] Here we assume that $m > 0$ and the all zero modes have positive chirality, i.e. $\Gamma\psi_\xi^i(x) = \psi_\xi^i(x)$, which means that only the m D0-branes are remained and all anti-D0-branes disappear after the tachyon condensation. However, this assumptions is not essential, even for cases with zero modes of both chiralities, as discussed in [21, 22].

for $U_\nu = \Omega_\nu e^{2\pi i \hat{p}_\nu R_\nu}$, we find an equivalence between the N D($2p$)-branes on the T^{2p} with the gauge field $A_\mu(x)$ and the m D0-branes on the *same* T^{2p} with the coordinates $\bar{X}^\mu$ (3.20). From the relation (2.3), the T-dual of the latter D0-branes is m D($2p$)-branes on the dual torus $\tilde{T}^{2p}$ with the gauge field $\tilde{A}_\mu(\xi)$. Note that

$$m = \int_{T^{2p}} \mathrm{Tr}\, e^{\frac{F}{2\pi}}, \quad N = \int_{\tilde{T}^{2p}} \mathrm{Tr}\, e^{\frac{\tilde{F}}{2\pi}}, \tag{3.24}$$

are followed from the index theorem.

Therefore, we find an equivalence between the N D($2p$)-branes on the T^{2p} with the gauge field $A_\mu(x)$ and the m D($2p$)-branes on the dual torus $\tilde{T}^{2p}$ with the gauge field $\tilde{A}_\mu(\xi)$ given by (3.21) using the Dirac zero modes (3.16).

The transition function for $\tilde{A}_\mu(\xi)$ is given by

$$\left(\tilde{\Omega}_\nu(\xi)\right)^i{}_j = \int_{\mathbf{R}^{2p}} d^{2p}x\, \Psi^i_\xi(x)^\dagger\, \Psi^j_{\xi'}(x) = \int_{T^{2p}} d^{2p}x'\, \psi^i_\xi(x')^\dagger\, \psi^{(\nu)}{}^j_\xi(x'), \tag{3.25}$$

where

$$\xi'_\mu = \xi_\mu + 2\pi\delta_{\mu\nu} L_\nu, \tag{3.26}$$

and

$$\psi^{(\nu)}{}^j_\xi(x) = e^{\frac{L_\nu x^\nu}{\alpha'}} \psi^j_{\xi'}(x), \tag{3.27}$$

which satisfies $\not{D}_\xi\, \psi^{(\nu)}{}^j_\xi(x) = 0$, thus a linear combinations of $\psi^i_\xi(x)$.

If we take $p = 2$ and $A_\mu(x)$ is anti-self dual, the formula (3.21) is indeed the Nahm transformation of [16, 17], which is a generalization of the formula given in [29] for the ADHM case. We note that the Nahm transformation can be viewed as a combination of the two different equivalences: (1) the T-dual and (2) the equivalence between the N D4-brane with A_μ and the m D0-branes with $\bar{X}^\mu$.

3.2 T-dual of the D0$-\overline{\text{D0}}$-brane pairs

Let us take the T-dual of the D($2p$)-brane solution on the torus, (3.1). Now we assume that the bundle on the D($2p$)-brane is trivial and $A_\mu(x) = \zeta_\mu/(2\pi\alpha')$, where ζ_μ is a constant. In this case, U_ν is just a translation operator. As we have seen, the Hilbert space of the Chan-Paton index is spanned by the spinors on the $\mathbf{R}^{2p}$ and any spinor $\Psi(x)$ on $\mathbf{R}^{2p}$ can be written as

$$\begin{aligned}\Psi(x) &= \int_0^{\frac{2\pi\alpha'}{R_1}} d\xi_1 \int_0^{\frac{2\pi\alpha'}{R_2}} d\xi_2 \cdots \int_0^{\frac{2\pi\alpha'}{R_{2p}}} d\xi_{2p} \sum_{n_\mu \in \mathbf{Z}} e^{i\frac{1}{2\pi\alpha'}\left(\xi_\mu + \frac{2\pi\alpha'}{R_\mu} n_\mu\right) x^\mu} \psi(\xi, n) \\ &= \int_0^{\frac{2\pi\alpha'}{R_1}} d\xi_1 \int_0^{\frac{2\pi\alpha'}{R_2}} d\xi_2 \cdots \int_0^{\frac{2\pi\alpha'}{R_{2p}}} d\xi_{2p} e^{i\frac{1}{2\pi\alpha'}\xi_\mu x^\mu}\, \psi_\xi(x),\end{aligned} \tag{3.28}$$

which is just a Fourier transformation with the momentum $p_\mu = \frac{1}{2\pi\alpha'}\left(\xi_\mu + \frac{2\pi\alpha'}{R_\mu} n_\mu\right)$. Here we defined $\psi_\xi(x) = \sum_{n_\mu \in \mathbf{Z}} e^{i\frac{1}{2\pi\alpha'}\frac{2\pi\alpha'}{R_\mu} n_\mu x^\mu} \psi(\xi, n)$ which is periodic, namely, $U_\nu \psi_\xi(x) = \psi_\xi(x)$ and $\psi(\xi, n)$ is a constant spinor. Then,

$$\Psi_{\xi,n}(x) = e^{i\frac{1}{2\pi\alpha'}\left(\xi_\mu + \frac{2\pi\alpha'}{R_\mu} n_\mu\right) x^\mu} \psi, \tag{3.29}$$

is a basis of the Hilbert space labeled by $\{\xi, n\}$ and the spinor index of a constant spinor ψ. We note that $\Psi_{\xi,n}(x)$ is an eigen state of U_ν,

$$U_\nu \Psi_{\xi,n}(x) = e^{i\frac{R_\nu}{\alpha'}\xi_\nu} \Psi_{\xi,n}(x) \tag{3.30}$$

and also an eigen state of the tachyon $T = \lim_{u\to\infty} u\, \not{D}$,

$$\not{D}\ \Psi_{\xi,n}(x) = \frac{1}{2\pi\alpha'}\Gamma^\mu \left(\xi_\mu + \frac{2\pi\alpha'}{R_\mu} n_\mu - \zeta_\mu\right)\ \Psi_{\xi,n}(x). \tag{3.31}$$

In this basis, $X^\mu = \hat{x}^\mu$ is represented as $2\pi\alpha' i\frac{\partial}{\partial \xi^\mu}$. This means that the gauge fields of the D($2p$)-branes and anti-D($2p$)-branes in the T-dual picture vanish.

Now we expect that the T-dual of the tachyon will be given by the tachyon in the above basis of the Chan-Paton bundle since we regard the torus as the orbifold of $\mathbf{R}^{2p}$. Therefore, in the T-dual picture, this system is equivalent to infinitely many pairs of D($2p$)-brane and anti-D($2p$)-brane, labeled by $\{n_\mu\} \in \mathbf{Z}^{2p}$, on the dual torus $\tilde{T}^{2p}$. The tachyon condensation is given by

$$\tilde{T}(\xi) = \tilde{u}\Gamma^\mu \left(\xi_\mu - \zeta_\mu + 2\pi L_\mu n_\mu\right), \tag{3.32}$$

which is diagonal in n_μ and $\tilde{A}_\mu = 0$. Here we defined $\tilde{u} = \frac{u}{2\pi\alpha'}$. We interpreted that the $\{\xi_\mu\}$ parameterize the world volume of the pairs of D2$-\overline{\text{D2}}$-branes, on the other hand, $\{n_\mu\}$ are the Chan-Paton indices. ζ_μ is the location of the a solitonic D0-branes on the dual torus $\tilde{T}^{2p}$.

Since $\Psi_{\xi',n} = \Psi_{\xi,n'}$, where $\xi'_\mu = \xi_\mu + 2\pi\delta_{\mu\nu}L_\nu$ and $n'_\mu = n_\mu + \delta_{\mu\nu}$, the transition function of the infinitely many pairs of D($2p$)-branes and anti-D($2p$)-branes in this T-dual picture, is given by

$$\tilde{\Omega}_\nu = U_{n_\mu \to n_\mu + \delta_{\mu,\nu}}, \tag{3.33}$$

where $U_{n_\mu \to n_\mu+\delta_{\mu,\nu}}$ is the unitary operator which maps $\Psi_{\xi,n}$ to $\Psi_{\xi,n'}$. Thus the tachyon (3.32) is a consistent configuration on the dual torus although it is not periodic.[9]

We note that the configuration (3.8) of the D0$-\overline{\text{D0}}$ pairs is more convenient than its T-dual configuration (3.32) of D($2p$)-anti-D($2p$) pairs, especially, for a configuration with a non-trivial $A_\mu(x)$. For a non-trivial $A_\mu(x)$, from a spinor $\psi_{\xi,n}(x)$ stisfying

$$U_\nu \psi_{\xi,n}(x) = \psi_{\xi,n}(x), \quad \not{D}_\xi \psi_{\xi,n}(x) = E_{\xi,n}\psi_{\xi,n}(x), \tag{3.34}$$

a basis is given by

$$\Psi_{\xi,n}(x) = e^{i\frac{1}{2\pi\alpha'}\xi_\mu x^\mu} \psi_{\xi,n}(x), \tag{3.35}$$

where $\Psi_{\xi,n}(x)$ is an eigen state of U_ν and $\not{D}$. Then, the tachyon configuration of D($2p$)-anti-D($2p$) pairs on $\tilde{T^{2p}}$ is implicitly given by

$$\tilde{T}(\xi) = u\, E_{\xi,n} \tag{3.36}$$

in this basis.

[9]We thank Koji Hashimoto for suggesting this solution.

Let us take the large radius limit of the torus or the T-dual torus. If we take the $L_\mu \to \infty$, then only the pair of D($2p$)-brane and anti-D($2p$)-brane with $n_\mu = 0$ will remain and the configuration (3.32) becomes

$$\tilde{T}(\xi) = \tilde{u}\gamma^\mu \left(\xi_\mu - \zeta_\mu\right), \tag{3.37}$$

which is just the Atiyah-Bott-Shapiro solution [5 − 7] for the decent relation. On the other hand, if we take the $R_\mu \to \infty$, the original infinitely many D0$-\overline{\text{D0}}$-brane pairs on T^r become those on $\mathbf{R}^r$ and the solution (3.1) is same as the solution for the ascent relation found in [12, 13]. Thus we can say that on the torus the two solutionsfor the decent relation (3.37) and the ascent relation (3.1) are T-dual each other.

Finally, we will comment on the classification of the D-branes by the K-theory. The configuration (3.37) for the decent relation is related to the K-theory (using the infinitely many D9$-\overline{\text{D9}}$ pairs [30, 31]). On the other hand, (3.1) represnts the (analytic) K-homology class in [13]. Therefore, we expext that the T-dual maps the K-theory to the analytic K-homology. However, the winding modes are neglected to obtain the analytic K-homology by assuming the size of the compactified manifold is very large in [13] although the winding modes are important for the T-dual picture. The duality of the KK-theory discussed in [32] will be important to study the role of the widing modes. It would be interesting to investigate the relation to it further.

4. Concluding remarks

In this paper, we found an exact solution, with a nonzero net D-brane charge, of the tachyon condesation in the boundary string field theory of brane-anti-brane pairs on torus. The Nahm-transformation of the instantons was derived from this tachyon condensation. We also found the T-dual configutration of this.

There are several interesting future directions. Since our method is not restricted to the instanton (i.e. $p = 2$) case, it will be interesting to study the Nahm transformation for D0-D8 or D0-D6 cases. Morevoer, the BPS properties are not (explicitly) assumed in this paper. Therefore, the non-BPS cases, for exampole models dicussed in [33 − 35] are also covered in thie paper. To extend our result to other orbifolds, like ALE spaces, are also interesting.

In this paper, we do not explicitly use the boundary state formalism though we believe the exact solutions in the boundary string field theory can be mapped to the boudnary state. (The marginal deformation case [15] was studied in [36].) It would be desired to do it explicitly.

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References

[1] A. Sen, *Tachyon condensation on the brane antibrane system*, *JHEP* **08** (1998) 012 [hep-th/9805170]; *Descent relations among bosonic D-branes*, *Int. J. Mod. Phys.* **A 14** (1999) 4061 [hep-th/9902105]; *Non-BPS states and branes in string theory*, hep-th/9904207; *Universality of the tachyon potential*, *JHEP* **12** (1999) 027 [hep-th/9911116].

[2] A. Sen, *Tachyon dynamics in open string theory*, *Int. J. Mod. Phys.* **A 20** (2005) 5513 [hep-th/0410103].

[3] D. Kutasov, M. Mariño and G.W. Moore, *Some exact results on tachyon condensation in string field theory*, *JHEP* **10** (2000) 045 [hep-th/0009148].

[4] A.A. Gerasimov and S.L. Shatashvili, *On exact tachyon potential in open string field theory*, *JHEP* **10** (2000) 034 [hep-th/0009103].

[5] D. Kutasov, M. Mariño and G.W. Moore, *Remarks on tachyon condensation in superstring field theory*, hep-th/0010108.

[6] P. Kraus and F. Larsen, *Boundary string field theory of the DD-bar system*, *Phys. Rev.* **D 63** (2001) 106004 [hep-th/0012198].

[7] T. Takayanagi, S. Terashima and T. Uesugi, *Brane-antibrane action from boundary string field theory*, *JHEP* **03** (2001) 019 [hep-th/0012210].

[8] T. Asakawa, S. Sugimoto and S. Terashima, *Exact description of D-branes via tachyon condensation*, *JHEP* **02** (2003) 011 [hep-th/0212188]; *Exact description of D-branes in K-matrix theory*, *Prog. Theor. Phys. Suppl.* **152** (2004) 93 [hep-th/0305006].

[9] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [hep-th/0511286].

[10] S. Teraguchi, *Reformulation of boundary string field theory in terms of boundary state*, *JHEP* **02** (2007) 017 [hep-th/0610171].

[11] A. Ishida and S. Teraguchi, *On the general action of boundary (super)string field theory*, *JHEP* **07** (2008) 020 [arXiv:0805.1826].

[12] S. Terashima, *A construction of commutative D-branes from lower dimensional non-BPS D-branes*, *JHEP* **05** (2001) 059 [hep-th/0101087].

[13] T. Asakawa, S. Sugimoto and S. Terashima, *D-branes, matrix theory and K-homology*, *JHEP* **03** (2002) 034 [hep-th/0108085].

[14] T. Asakawa, S. Sugimoto and S. Terashima, *D-branes and KK-theory in type-I string theory*, *JHEP* **05** (2002) 007 [hep-th/0202165].

[15] J. Majumder and A. Sen, *Vortex pair creation on brane-antibrane pair via marginal deformation*, *JHEP* **06** (2000) 010 [hep-th/0003124].

[16] P.J. Braam and P. van Baal, *Nahm's transformation for instantons*, *Commun. Math. Phys.* **122** (1989) 267.

[17] H. Schenk, *On a generalized fourier transform of instantons over flat tori*, *Commun. Math. Phys.* **116** (1988) 177.

[18] K. Hori, *D-branes, T-duality and index theory*, *Adv. Theor. Math. Phys.* **3** (1999) 281 [hep-th/9902102].

[19] M.R. Douglas, *Gauge fields and D-branes*, *J. Geom. Phys.* **28** (1998) 255 [hep-th/9604198].

[20] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, *Nucl. Phys.* **B 503** (1997) 220 [hep-th/9608163].

[21] K. Hashimoto and S. Terashima, *Stringy derivation of Nahm construction of monopoles*, *JHEP* **09** (2005) 055 [hep-th/0507078].

[22] K. Hashimoto and S. Terashima, *ADHM is tachyon condensation*, *JHEP* **02** (2006) 018 [hep-th/0511297].

[23] S. Terashima, *Noncommutativity and tachyon condensation*, *JHEP* **10** (2005) 043 [hep-th/0505184].

[24] S. Terashima, *Supertubes in matrix model and DBI action*, *JHEP* **03** (2007) 075 [hep-th/0701179].

[25] W. Taylor, *D-brane field theory on compact spaces*, *Phys. Lett.* **B 394** (1997) 283 [hep-th/9611042].

[26] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[27] I. Ellwood, *Relating branes and matrices*, *JHEP* **08** (2005) 078 [hep-th/0501086].

[28] M.F. Atiyah and I.M. Singer *The index of elliptic operators.* 4, *Ann. Math.* **93** (1971) 119; *The index of elliptic operators.* 5, *Ann. Math.* **93** (1971) 139.

[29] E. Corrigan and P. Goddard, *Construction of instanton and monopole solutions and reciprocity*, *Ann. Phys. (NY)* **154** (1984) 253.

[30] J.A. Harvey and G.W. Moore, *Noncommutative tachyons and K-theory*, *J. Math. Phys.* **42** (2001) 2765 [hep-th/0009030].

[31] E. Witten, *Overview of K-theory applied to strings*, *Int. J. Mod. Phys.* **A 16** (2001) 693 [hep-th/0007175].

[32] J. Brodzki, V. Mathai, J. Rosenberg and R.J. Szabo, *Noncommutative correspondences, duality and D-branes in bivariant K-theory*, arXiv:0708.2648.

[33] W. Taylor, *Adhering* 0*-branes to* 6*-branes and* 8*-branes*, *Nucl. Phys.* **B 508** (1997) 122 [hep-th/9705116].

[34] M. Hamanaka and H. Kajiura, *Gauge fields on tori and T-duality*, *Phys. Lett.* **B 551** (2003) 360 [hep-th/0208059].

[35] R. Wimmer, *D*0*-D*4 *brane tachyon condensation to a BPS state and its excitation spectrum in noncommutative super Yang-Mills theory*, *JHEP* **05** (2005) 022 [hep-th/0502158].

[36] M. Naka, T. Takayanagi and T. Uesugi, *Boundary state description of tachyon condensation*, *JHEP* **06** (2000) 007 [hep-th/0005114].

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